

# The Ground Ring of $N = 2$ Minimal String Theory

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## Abstract

We study the  $\mathcal{N} = 2$  string theory or the  $\mathcal{N} = 4$  topological string on the deformed CHS background. That is, we consider the  $\mathcal{N} = 2$  minimal model coupled to the  $\mathcal{N} = 2$  Liouville theory. This model describes holographically the topological sector of Little String Theory. We use degenerate vectors of the respective  $\mathcal{N} = 2$  Verma modules to find the set of BRST cohomologies at ghost number zero—the ground ring, and exhibit its structure. Physical operators at ghost number one constitute a module of the ground ring, so the latter can be used to constrain the S-matrix of the theory. We also comment on the inequivalence of BRST cohomologies of the  $\mathcal{N} = 2$  string theory in different pictures.

# 1 Introduction

The ground ring proved to be a powerful tool in constraining the dynamics of non-critical bosonic and  $\mathcal{N} = 1$  fermionic string theories. In [1] Lian and Zuckerman constructed BRST cohomologies at various ghost numbers. A particularly interesting subset of these BRST invariant operators is a set of operators of ghost number zero. Under the operator product these operators form a ring, which is called the ground ring. Its importance was first realized in [2], where the relations to the symmetries of the theory were revealed. More recently the relations between the ground ring and the D-branes of the theory were understood [3]. In particular it was shown that boundary states based on the ZZ branes [4] turn out to be the eigenvectors of the ground ring generators. Moreover, the corresponding eigenstates label the singular points in the moduli space of boundary couplings.

An interesting set of string theories is obtained by gauging  $\mathcal{N} = 2$  superconformal symmetry on the worldsheet. The resulting model is called the  $\mathcal{N} = 2$  string [5, 6]. The ground ring of the critical  $\mathcal{N} = 2$  string was studied in [7]. In this paper we study the ground ring of the  $\mathcal{N} = 2$  minimal string theory. This model is obtained by coupling the  $SU(2)/U(1)$  superparafermions to the  $\mathcal{N} = 2$  Liouville theory, or equivalently to the  $\mathcal{N} = 2$   $SL(2)/U(1)$  supersymmetric coset [8]. We will use these equivalent descriptions interchangeably in the paper. One of the main motivations for our work is better understanding of the topological sector of Little String Theory (LST). LST appears in various decoupling limits of string theories which contain NS5 branes or singularities. An interesting property of LST is that while being a non-local theory it does not contain gravity. This theory was extensively studied (for recent review and further references see [9, 10]) using the holographically dual description [11], and many interesting features have been unveiled [12–16]. The simplest of LSTs are  $5 + 1$  dimensional theories with sixteen supercharges. They arise from the decoupling limit of  $k$  type *IIA* or type *IIB* NS5-branes in flat space. The holographic description of these theories is given by closed strings in the near horizon geometry of NS5 branes—the CHS background [17]. Unfortunately, string theory in this background is strongly coupled due to the presence of the linear dilaton. One way to avoid this problem is to consider the theory at a non-singular point in the moduli space. The simplest such configuration corresponds to NS5 branes distributed on a circle. In this case the CHS background gets deformed into [18]

$$\mathbb{R}^{5,1} \times \left( \frac{SU(2)}{U(1)} \times \frac{SL(2)}{U(1)} \right) / \mathbb{Z}_k, \quad (1.1)$$

where the  $\mathbb{Z}_k$  orbifolding ensures the R-charge integrality, *i.e.* imposes the GSO projection.

String theory in the CHS background (and its deformation) enjoys  $\mathcal{N} = 4$  worldsheet

supersymmetry, hence one can define  $\mathcal{N} = 4$  topological string theory in this background, which is equivalent to the  $\mathcal{N} = 2$  string [19]. This topological string captures the sector of string theory in the background (1.1), which is protected by supersymmetry. This is similar to the description of the BPS sector of type II string theory compactified on Calabi-Yau threefold by the  $\mathcal{N} = 2$  topological string. One expects that the  $\mathcal{N} = 2$  string should holographically describe the topological version of Little String Theory. In [20] the validity of this proposal was checked by an explicit calculation of  $F^4$  coupling, where  $F$  is the Abelian gauge field in the low-energy theory, using Heterotic/Type II duality.

One can use the ground ring to effectively compute amplitudes of physical operators in the  $\mathcal{N} = 2$  string theory. Indeed, BRST invariant operators at ghost number one form a module of the ground ring and one can use this fact to express correlation functions of these operators in terms of the structure coefficients of the ground ring. Let us label the physical operators of the theory  $\mathcal{T}$  by multi-index  $[I]$ . Then  $\mathcal{T}_{[I]} = \mathcal{O}_{[I]} \mathcal{T}_{[0]}$ , where  $\mathcal{O}_{[I]}$  is a ground ring element and we can write

$$\langle \mathcal{T}_{[I]} \mathcal{T}_{[J]} \mathcal{T}_{[K]} \rangle = N_{IJK} \langle \mathcal{T}_{[0]} \mathcal{T}_{[0]} \mathcal{T}_{[0]} \rangle. \quad (1.2)$$

In this expression  $N_{IJK}$  are the structure constants of the ground ring. Unfortunately for the bosonic and  $\mathcal{N} = 1$  string the existence of contact terms [21] makes simple generalization of this result to four and higher point amplitudes hard. This is because one has to integrate over the worldsheet position of at least one of the operators. Hence, the explicit knowledge of the contact terms is required. It is plausible that bigger symmetry of the  $\mathcal{N} = 2$  string theory will allow one to understand the structure of contact terms, and possibly to generalize this method to four and higher-point amplitudes.

This paper is organized as follows. In Section 2 we define the  $\mathcal{N} = 2$  Minimal String Theory and provide necessary details.

In Section 3 the general construction of ground ring elements is presented. It is similar to that employed in bosonic and  $\mathcal{N} = 1$  fermionic minimal string theories. We establish that there is a one to one correspondence between the f-series degenerate vectors of Verma modules of  $SU(2)/U(1)$  and  $SL(2)/U(1)$  supercosets and ground ring elements in the  $(-1, -1)$  picture of the  $\mathcal{N} = 2$  string. Ground ring elements are labeled by the level at which the null vector appears in the corresponding superconformal family and a certain quantum number  $m$  which is related to the R-charge of the  $SU(2)/(1)$  coset.

In Section 4 we explicitly construct ground ring elements based on the level one degenerate operators in the  $SU(2)/U(1)$  and  $SL(2)/U(1)$  supercosets and show that they generate a  $\mathbb{Z}_k$  subring. This subring acts on the elements of ground ring by shifting  $m$ .

In Section 5 we consider degenerate vectors which give rise to null vectors at level two. We find the explicit form of null vectors and use them to construct the level two ground

ring elements. We argue that they are the generators of the ground ring.

In Section 6 ground ring operators based on g-series degenerate vectors are discussed. We show that the operators responsible for the spectral flow symmetry of the theory belong to this category along with the unity operator. These operators have total picture 0. We argue that these are the only independent ground ring elements at this picture based on g-series.

We conclude with the discussion of our results and directions for future research.

Appendix A provides necessary information about the  $\mathcal{N} = 2$  superconformal ghosts. In Appendix B we present some useful facts about  $SU(2)/U(1)$  and  $SL(2)/U(1)$  supercosets and derive the fusion rules of the degenerate operators at level one. In Appendix C we give a detailed description of the fusion of the level one ground ring operators. In Appendix D we give an example of a kernel and a co-kernel of a picture-raising operator, hence illustrating the inequivalence of different pictures in the  $\mathcal{N} = 2$  string theory.

## 2 $\mathcal{N} = 2$ minimal strings

The worldsheet matter content of the  $\mathcal{N} = 2$  minimal string theory is given by the product of two  $\mathcal{N} = 2$  superconformal cosets  $SL(2)/U(1) \times SU(2)/U(1)$ . The level of both cosets is a positive integer  $k$ . The corresponding (coset) central charges are

$$c_{SU} = 3 \left( 1 - \frac{2}{k} \right), \quad c_{SL} = 3 \left( 1 + \frac{2}{k} \right). \quad (2.1)$$

The  $\mathcal{N} = 2$  superconformal algebra reads

$$\begin{aligned} [L_m, G_r^\pm] &= \left( \frac{m}{2} - r \right) G_{m+r}^\pm, \\ [L_m, J_m] &= -n J_{m+n}, \\ \{G_r^+, G_s^-\} &= 2L_{r+s} + (r-s)J_{r+s} + \frac{c}{3} \left( r^2 - \frac{1}{4} \right) \delta_{r,-s}, \\ \{G_r^+, G_s^-\} &= \{G_r^-, G_s^-\} = 0, \\ [J_n, G_r^\pm] &= \pm G_{r+n}^\pm, \\ [J_m, J_n] &= \frac{c}{3} m \delta_{m,-n}, \end{aligned} \quad (2.2)$$

where  $L_n$  are the Virasoro algebra generators,  $G_r^\pm$  are the modes of the supercurrents ( $r \in \mathbb{Z} + 1/2$  in the Neveu-Schwarz and  $r \in \mathbb{Z}$  in the Ramond sectors) and  $J_n$  are modes of the  $U(1)$  R-current.

Upon gauging the world-sheet  $\mathcal{N} = 2$  superconformal algebra one introduces the usual fermionic  $(b, c)$  conformal ghost system, a pair of bosonic  $(\beta^\pm, \gamma^\mp)$  systems and an additional fermionic  $(\tilde{b}, \tilde{c})$  system with weights  $(0, 1)$  that arises when gauging the R-current.

Our conventions regarding the ghost systems are collected in appendix A. The BRST current takes the form

$$j_{BRST} = cT + \eta_- e^{\phi_-} G^+ + \eta_+ e^{\phi_+} G^- + \tilde{c} J^m + \frac{1}{2} [cT^{gh} + \eta_- e^{\phi_-} G_{gh}^+ + \eta_+ e^{\phi_+} G_{gh}^- + \tilde{c} J^{gh}], \quad (2.3)$$

where  $T^{gh}$ ,  $G_{gh}^\pm$ ,  $J^{gh}$  are the ghost  $\mathcal{N} = 2$  currents whose explicit expressions are given in (A.1),  $\eta^\pm$  and  $\phi_\pm$  are bosonized superghosts (see appendix A). The total central charge of the  $\mathcal{N} = 2$  ghost system equals -6; this matches the total central charge of the matter theory (2.1).

The chiral BRST operator

$$Q_{BRST} = \frac{1}{2\pi i} \oint dz j_{BRST}, \quad (2.4)$$

commutes with the ghost number current

$$j_{gh} = -bc - \tilde{b}\tilde{c} + \eta^+ \xi^- + \eta^- \xi^+, \quad (2.5)$$

and the two picture number currents

$$j_{\pi+} = -\eta^+ \xi^- - \partial\phi_+; \quad j_{\pi-} = -\eta^- \xi^+ - \partial\phi_-. \quad (2.6)$$

The corresponding cohomology groups are thus labeled by the ghost number and the picture numbers  $(\Pi_+, \Pi_-)$ .

One can define two picture raising operators

$$PCO^\pm = \{Q, \xi^\pm\} = c\partial\xi^\pm + e^{\phi^\mp} (G^\pm - 2\eta^\pm e^{\phi^\pm} b \pm 2\partial(\eta^\pm e^{\phi^\pm})\tilde{b} \pm \eta^\pm e^{\phi^\pm} \partial\tilde{b}). \quad (2.7)$$

It is known [7] that unlike in the  $\mathcal{N} = 1$  superstrings, the picture raising operators (2.7) are not isomorphisms of the absolute cohomology groups at different pictures (see appendix D for explicit examples of the  $PCO^\pm$  kernel and cokernel elements). This complicates the analysis of the cohomologies.

The  $SU(2)/U(1)$  superparafermion primaries  $\mathcal{V}_{j,m}$  are labeled by  $j \in \frac{1}{2}\mathbb{Z}$ ,  $0 \leq j \leq \frac{k-2}{2}$  and a number  $m \in \{-j, -j+1, \dots, j-1, j\}$ . In the NS sector these primaries have conformal dimensions

$$\Delta[\mathcal{V}_{j;m}] = \frac{j(j+1)}{k} - \frac{m^2}{k} \quad (2.8)$$

and the  $R$ -charge

$$q(\mathcal{V}_{j,m}) = -\frac{2m}{k}. \quad (2.9)$$

The  $SL(2)/U(1)$   $\mathcal{N} = 2$  primaries are denoted by  $\mathcal{V}'_{h,m}$ . They are labeled by two real numbers  $h$  and  $m$ . The latter number takes the values  $m \in h + \mathbb{Z}_+$  for a lowest weight

representation,  $m \in -h + \mathbb{Z}_-$  for a highest weight one and  $m \in -h, -h+1, \dots, h-1, h$  for a finite one. The conformal dimension reads

$$\Delta[\mathcal{V}'_{h,m}] = -\frac{h(h-1)}{k} + \frac{m^2}{k} \quad (2.10)$$

while the R-charge is given by

$$q(\mathcal{V}'_{h,m}) = \frac{2m}{k}. \quad (2.11)$$

### 3 Construction of ground ring elements

As explained above, the ground ring consists of physical states at ghost number 0. In this section we describe the construction of nontrivial elements of the ground ring following [22], where (p,q) minimal models coupled to Liouville theory were studied. The minimal models contain a large set of null vectors in the Verma module; each of them can be dressed with a null vector in the Liouville theory to produce a physical state at ghost number 1. It has been argued [22] that these null vectors should be set to zero in the physical theory. Hence, the ghost number 1 physical state described above should be set to zero. Its BRST pre-image is a non-trivial physical state at ghost number 0 [22], otherwise known as a ground ring element.

Let us review the structure of the null vectors in  $SU(2)/U(1)$  and  $SL(2)/U(1)$ , following [23] (see also [24]). We first consider  $SU(2)/U(1)$  at level  $k$  ( $SL(2)/U(1)$  expressions are obtained by the substitution  $k \rightarrow -k$ ). Here we only discuss the NS sector. The R sector can be obtained from it by the spectral flow. The ground ring elements which realize this operation are described below.

The corresponding Kac determinant has two series of zeroes specified by two functions  $f_{r,s}$  and  $g_l$  [23]:

$$f_{r,s} = 2(\tilde{c} - 1)\Delta - q^2 - \frac{1}{4}(\tilde{c} - 1)^2 + \frac{1}{4}[(\tilde{c} - 1)r + 2s]^2, \quad (3.1)$$

and

$$g_l = 2\Delta - 2lq + (\tilde{c} - 1)(l^2 - \frac{1}{4}), \quad (3.2)$$

where

$$\tilde{c} - 1 = \pm \frac{2}{k}, \quad (3.3)$$

where the  $- (+)$  sign is for  $SU(2)/U(1)$  ( $SL(2)/U(1)$ ). The null states are at the level  $rs$  for the  $f$ -series and at the level  $|l|$  for the  $g$ -series states. The R-charge of the null states differs from that of the corresponding primaries by zero in the case of  $f$ -series, and by  $\pm 1 = \text{sign}(l)$  for the  $g$ -series.

We want to construct  $(-1, -1)$  picture physical states at ghost number 1. A natural ansatz would be

$$ce^{-\phi_+}e^{-\phi_-}\{-rs\}\mathcal{V}_{j,m_{su}}\mathcal{V}'_{h,m_{sl}}, \quad (3.4)$$

where  $\{-rs\}$  stands for a raising operator at level  $rs$  acting on the  $SU(2)$  superparafermion primary  $\mathcal{V}_{j,m_{su}}$  and the  $SL(2)$  superparafermion primary  $\mathcal{V}'_{h,m_{sl}}$ . The condition of vanishing R-charge implies  $m_{su} = m_{sl}$ , so that  $q_{su} = -q_{sl}$ . According to (3.1) and (3.2), the dimension of the matter part in (3.4) is zero for  $f$ -series and  $|l|$  for the  $g$ -series. Hence, it is the  $f$ -series which is well suited for the construction of physical states of the type (3.4). Below we will identify the quantum numbers which correspond to the zeroes of (3.1). We will discuss the  $g$ -series later in the paper.

The quantum numbers of degenerate states whose conformal dimension and R-charge satisfy (3.1) can be obtained with the help of (2.10), (2.11) for  $SL(2)/U(1)$  theory and (2.8), (2.9) for  $SU(2)/U(1)$ . The result for the latter is  $j_{r,s} = \frac{ks-1-r}{2}$ . Since  $j$  is restricted to lie between 0 and  $(k-2)/2$ , we must set  $s = 1$ . Hence, degenerate states in the supersymmetric  $SU(2)/U(1)$  which have a null descendant at level  $s$  are parameterized by

$$j_r = \frac{k-1-r}{2}, \quad r = 1, \dots, k-1; \quad m = -j_r, -j_r+1, \dots, j_r. \quad (3.5)$$

Corresponding degenerate states in the supersymmetric  $SL(2)/U(1)$  are labeled by

$$h_r = -\frac{k-1+r}{2}, \quad r = 1, \dots, k-1; \quad m = h_r, h_r+1, \dots, -h_r. \quad (3.6)$$

We now describe the construction of a ground ring operator based on the  $f$ -series degenerate operators  $\mathcal{V}_{j_r,m}$ ,  $\mathcal{V}'_{h_r,m}$ , with  $j_r$  and  $h_r$  given by (3.5) and (3.6), respectively. The physical state operator at ghost number 1, which should be set to zero, has the following general form

$$\mathcal{O}' = ce^{-\phi_+}e^{-\phi_-}(\lambda\{-rs, su\} + \lambda'\{-rs, sl\})\mathcal{V}_{j_r,m}\mathcal{V}'_{h_r,m}, \quad (3.7)$$

where  $\lambda$  and  $\lambda'$  are arbitrary coefficients. The corresponding state is equal to zero in the physical theory. The ground ring element  $\mathcal{O}$  is found by requiring

$$Q_{BRST}\mathcal{O} = \mathcal{O}'. \quad (3.8)$$

Below we consider  $r = 1$  and  $r = 2$  cases in detail. We find that the operator  $\mathcal{O}$  is determined up to a rescaling and an addition of BRST-trivial piece which corresponds to the choice of  $\lambda$  and  $\lambda'$  in (3.7). Note that  $\mathcal{O}$  is a ground ring element in the  $(-1, -1)$  picture.

The operators that will be useful in the following are [20, 25]

$$\mathcal{S}^+ \equiv e^{\tilde{b}c/2}e^{\phi_+/2-\phi_-/2}\mathcal{V}_{0,0}(R, -)\mathcal{V}'_{0,0}(R, -), \quad (3.9)$$

$$\mathcal{S}^- \equiv e^{-\tilde{b}c/2}e^{-\phi_+/2+\phi_-/2}\mathcal{V}_{0,0}(R, +)\mathcal{V}'_{0,0}(R, +). \quad (3.10)$$

These operators are in the BRST cohomology of the  $\mathcal{N} = 2$  string theory at ghost number 0, and hence are elements of the ground ring. They satisfy

$$\mathcal{S}^+ \mathcal{S}^- \sim 1, \quad (3.11)$$

and map NS sector states into R sector states, thereby providing an isomorphism between the two sectors of the theory. That is why it is sufficient to consider only NS sector.

By squaring (3.9) we obtain new ground ring operators

$$(\mathcal{S}^+)^2 \equiv e^{\tilde{b}c} e^{\phi_+ - \phi_-} \mathcal{V}_{\frac{k-2}{2}, \frac{k-2}{2}} \mathcal{V}'_{\frac{k+2}{2}, -\frac{k+2}{2}}, \quad (3.12)$$

$$(\mathcal{S}^-)^2 \equiv e^{-\tilde{b}c} e^{-\phi_+ + \phi_-} \mathcal{V}_{\frac{k-2}{2}, -\frac{k-2}{2}} \mathcal{V}'_{\frac{k+2}{2}, \frac{k+2}{2}}. \quad (3.13)$$

in the (1,-1) and (-1,1) pictures, respectively. It is not hard to see that these operators have zero dimension and R-charge. Their existence implies that BRST cohomologies at picture  $(\Pi_+, \Pi_-)$  and  $(\Pi_+ + n, \Pi_- - n)$ ,  $n \in \mathbb{Z}$  are isomorphic. There exist two picture raising operators (2.7) but they do not have an inverse in the  $\mathcal{N} = 2$  string<sup>1</sup>.

#### 4 Ground ring operators at level one

A null vector belonging to the superconformal family generated by  $|\Delta, q\rangle$  at the first level (f-series) is given by

$$|\chi\rangle = [(q-1)L_{-1} - (2\Delta+1)J_{-1} + G_{-1/2}^+ G_{-1/2}^-] |\Delta, q\rangle, \quad (4.1)$$

where  $\Delta$  and  $q$  satisfy

$$2(\tilde{c}-1)\Delta = q^2 - \tilde{c}. \quad (4.2)$$

In this expression  $\tilde{c} = c/3$  is the (rescaled) central charge of  $SL(2)/U(1)$  (upper sign) or  $SU(2)/U(1)$  (lower sign) cosets

$$\tilde{c} = 1 \pm \frac{2}{k}. \quad (4.3)$$

We would like to find a state such that the action of the BRST charge on it will yield a null vector of the form (4.1). The local operator corresponding to such a state will belong to the ground ring. More precisely, we would like to find an operator  $\mathcal{O}$  such that

$$\begin{aligned} \{Q_{BRST}, \mathcal{O}\} = \\ \lambda c[(q^{(su)} - 1)L_{-1}^{(su)} - (2\Delta^{su} + 1)J_{-1}^{(su)} + G_{-1/2}^{+(su)} G_{-1/2}^{-(su)}] \mathcal{V} \mathcal{V}' e^{-\phi_+} e^{-\phi_-} + \\ \lambda' c[(q^{(sl)} - 1)L_{-1}^{(sl)} - (2\Delta^{sl} + 1)J_{-1}^{(sl)} + G_{-1/2}^{+(sl)} G_{-1/2}^{-(sl)}] \mathcal{V} \mathcal{V}' e^{-\phi_+} e^{-\phi_-}. \end{aligned} \quad (4.4)$$

where  $\mathcal{V}$  and  $\mathcal{V}'$  are the degenerate operators at level one in  $SU(2)/U(1)$  and  $SL(2)/U(1)$  supercosets respectively and  $\lambda, \lambda'$  are proportionality coefficients. One can further simplify

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<sup>1</sup>An example of a kernel and a co-kernel of the picture raising operators is discussed in the Appendix D.



the expressions above, by noting that

$$\begin{aligned}\Delta^{(sl)} &= -1 - \Delta^{(su)} \equiv -1 - \Delta, \\ q^{(sl)} &= -q^{(su)} \equiv -q,\end{aligned}\tag{4.5}$$

that follows from the physical state condition on the operator  $\{Q, \mathcal{O}\}$ . These equations are consistent with the condition that degenerate operators in  $SU(2)/U(1)$  and  $SL(2)/U(1)$  cosets lie in the f-series at level one. We can solve the conditions (4.2), (4.5) explicitly to obtain

$$\mathcal{V}\mathcal{V}' = \mathcal{V}_{\frac{k-2}{2},m} \mathcal{V}'_{-\frac{k}{2},m} \equiv \mathcal{V}_m^{(1)} \mathcal{V}_m'^{(1)}\tag{4.6}$$

with

$$-\frac{k-2}{2} \leq m \leq \frac{k-2}{2}.\tag{4.7}$$

To find the operator  $\mathcal{O}$  we start by considering the following ansatz

$$\begin{aligned}\mathcal{O} &= e^{\tilde{\alpha}\tilde{b}\tilde{c}} \mathcal{V}_m^{(1)} \mathcal{V}_m'^{(1)} e^{-\phi_+} e^{-\phi_-} + \\ &\quad \partial\xi^- c(A_m^{(su)} G_{-1/2}^{+(su)} + A_m^{(sl)} G_{-1/2}^{+(sl)}) \mathcal{V}_m^{(1)} \mathcal{V}_m'^{(1)} e^{-2\phi_+} e^{-\phi_-} + \\ &\quad c\partial\xi^+ (B_m^{(su)} G_{-1/2}^{-(su)} + B_m^{(sl)} G_{-1/2}^{-(sl)}) \mathcal{V}_m^{(1)} \mathcal{V}_m'^{(1)} e^{-\phi_+} e^{-2\phi_-}.\end{aligned}\tag{4.8}$$

Acting by  $Q_{BRST}$  we find the following independent equations on the coefficients

$$\begin{aligned}B_m^{(sl)} &= A_m^{(su)} \equiv A_m, \\ A_m^{(sl)} &= B_m^{(su)} \equiv B_m, \\ B_m - A_m &= \lambda, \\ 1 + 2A_m &= \lambda(q_m - 1), \\ \alpha &= -(2\Delta_m + 1)\lambda.\end{aligned}\tag{4.9}$$

Solving these equations we find one parameter family of solutions

$$\begin{aligned}2A_m &= \lambda(q_m - 1) - 1, \\ 2B_m &= \lambda(q_m + 1) - 1, \\ \alpha &= -\lambda(2\Delta_m + 1).\end{aligned}\tag{4.10}$$

At first sight this result seems to be puzzling, since we expected to get a single solution for a fixed  $m$ . To resolve this puzzle we note that  $Q\mathcal{O}_{\lambda=0}$  is identically zero. So it seems that we found two “physical” operators instead of one. It turns out, however, that  $\mathcal{O}_{\lambda=0}$  is actually BRST exact. One can show that

$$\mathcal{O}_{\lambda=0} = \{Q, \frac{1}{2} c\partial\xi^+ \partial\xi^- \mathcal{V}_m^{(1)} \mathcal{V}_m'^{(1)} e^{-2\phi_+} e^{-2\phi_-}\}.\tag{4.11}$$

(A fast way to see that  $\mathcal{O}_{\lambda=0}$  is cohomological to zero is by checking  $PCO^+\mathcal{O} = 0$ .)

Hence indeed we obtain a single non-trivial ground ring operator corresponding to the given degenerate state. One can conveniently fix the gauge by considering  $\mathcal{O}_\lambda - \mathcal{O}_{\lambda=0}$  as

a representative of the ground ring. After an appropriate rescaling we obtain ground ring operators

$$\begin{aligned}\mathcal{O}_m^{(1)} = & \tilde{b}c\mathcal{V}\mathcal{V}'e^{-\phi_+}e^{-\phi_-} + \\ & \partial\xi^-c(A'_mG_{-1/2}^{+(su)} + B'_mG_{-1/2}^{+(sl)})\mathcal{V}_m^{(1)}\mathcal{V}'_m^{(1)}e^{-2\phi_+}e^{-\phi_-} + \\ & c\partial\xi^+(B'_mG_{-1/2}^{-(su)} + A'_mG_{-1/2}^{-(sl)})\mathcal{V}_m^{(1)}\mathcal{V}'_m^{(1)}e^{-\phi_+}e^{-2\phi_-},\end{aligned}\quad (4.12)$$

where

$$\begin{aligned}A'_m &= -\frac{(q_m-1)}{2(2\Delta_m+1)} = \frac{1}{k-2m}, \\ B'_m &= -\frac{(q_m+1)}{2(2\Delta_m+1)} = -\frac{1}{k+2m}.\end{aligned}\quad (4.13)$$

For certain applications it is convenient to consider these operators at higher pictures. In particular we would like to apply  $PCO^+$  to the operator (4.12)

$$\begin{aligned}PCO^+\mathcal{O}_m^{(1)} = & \tilde{b}cG_{-1/2}^+\mathcal{V}_m^{(1)}\mathcal{V}'_m^{(1)}e^{-\phi_+} + \partial\xi^-\frac{c}{(2\Delta+1)}e^{-2\phi_+}G_{-1/2}^{+(sl)}G_{-1/2}^{+(su)}\mathcal{V}_m^{(1)}\mathcal{V}'_m^{(1)} + \\ & 2(A'_mG_{-1/2}^{+(su)} + B'_mG_{-1/2}^{+(sl)})\mathcal{V}_m^{(1)}\mathcal{V}'_m^{(1)}e^{-\phi_+} + c\partial\xi^+e^{-\phi_+}e^{-\phi_-}\mathcal{V}_m^{(1)}\mathcal{V}'_m^{(1)}.\end{aligned}\quad (4.14)$$

The  $PCO^-$  for this operator reads

$$\begin{aligned}PCO^-\mathcal{O}_m^{(1)} = & -\tilde{b}cG_{-1/2}^-\mathcal{V}_m^{(1)}\mathcal{V}'_m^{(1)}e^{-\phi_-} + \partial\xi^+c\frac{1}{(2\Delta+1)}e^{-2\phi_-}G_{-1/2}^{-(sl)}G_{-1/2}^{-(su)}\mathcal{V}_m^{(1)}\mathcal{V}'_m^{(1)} + \\ & -2(B'_mG_{-1/2}^{-(su)} + A'_mG_{-1/2}^{-(sl)})\mathcal{V}_m^{(1)}\mathcal{V}'_m^{(1)}e^{-\phi_-} - c\partial\xi^-e^{-\phi_+}e^{-\phi_-}\mathcal{V}_m^{(1)}\mathcal{V}'_m^{(1)}.\end{aligned}\quad (4.15)$$

The  $PCO^+$  operator takes a particularly nice form when the  $\mathcal{V}_m^{(1)}$  is a chiral primary that is when  $m = 1 - k/2$ . Then we have

$$PCO^+\mathcal{O}_{1-k/2}^{(1)} = e^{-\phi_+}e^{-\tilde{b}c}G_{-1/2}^+\mathcal{V}_{1-k/2}^{(1)}\mathcal{V}'_{1-k/2}^{(1)} + c\partial\xi^+e^{-\phi_+}e^{-\phi_-}\mathcal{V}_{1-k/2}^{(1)}\mathcal{V}'_{1-k/2}^{(1)}. \quad (4.16)$$

We can compute the OPE of two level one ground ring operators  $PCO^+\mathcal{O}_{m_1}^{(1)}$  and  $PCO^-\mathcal{O}_{m_2}^{(1)}$  given by (4.14), (4.15). Thus the ground ring elements we are fusing are chosen in the gauge (4.12). To compute the OPE's of the matter operators we use formulas (B.12)-(B.16) and similar formulas for the  $SU(2)/U(1)$  coset.

After a straightforward computation we obtain that for  $m_1 + m_2 < 0$  we have the following product of cohomologies

$$PCO^+\mathcal{O}_{m_1}^{(1)}PCO^-\mathcal{O}_{m_2}^{(1)} = K_{m_1,m_2}\tilde{\mathcal{O}}_{m_3}^{(1)}, \quad (4.17)$$

where  $\tilde{\mathcal{O}}_{m_3}^{(1)}$  is the ground ring element given in (4.8) with  $m_3 = m_1 + m_2 + k/2$  and a  $\lambda = -1/(q_1 + q_2)$  fixed such that  $B_{m_3} = 0$  in (4.10). The numerical coefficient  $K_{m_1,m_2}$  in (4.17) is

$$K_{m_1,m_2} = -C_{m_1,m_2}\frac{(q_1 + q_2)^2[k(q_1 + q_2 - 2) - 2][k(q_1 + q_2 - 2) + 2]}{k^2(q_1 - 1)(q_2 - 1)} \quad (4.18)$$

where  $C_{m_1, m_2}$  is defined as an OPE coefficient

$$\mathcal{V}_{m_1}^{(1)} \mathcal{V}_{m_1}'^{(1)}(z_1) \mathcal{V}_{m_2}^{(1)} \mathcal{V}_{m_2}'^{(1)}(z_2) \sim \frac{C_{m_1, m_2}}{z_{12}^{\Delta_1 + \Delta_2 - \Delta_3}} G_{-1/2}^{+, su} \mathcal{V}_{m_3}^{(1)} G_{-1/2}^{-, sl} \mathcal{V}_{m_3}'^{(1)}. \quad (4.19)$$

For  $m_1 + m_2 > 0$  one obtains a result similar to (4.17) with an operator  $\tilde{\mathcal{O}}_{m_3}^{(1)}$ ,  $m_3 = m_1 + m_2 - k/2$  appearing on the RHS being in a different gauge and with a different proportionality coefficient

$$K'_{m_1, m_2} = C'_{m_1, m_2} \frac{(q_1 + q_2)^2 [k(q_1 + q_2 - 2) - 2] [k(q_1 + q_2 - 2) + 2]}{k^2(q_1 + 1)(q_2 + 1)}, \quad (4.20)$$

where  $C'_{m_1, m_2}$  is defined as follows

$$\mathcal{V}_{m_1}^{(1)} \mathcal{V}_{m_1}'^{(1)}(z_1) \mathcal{V}_{m_2}^{(1)} \mathcal{V}_{m_2}'^{(1)}(z_2) \sim \frac{C'_{m_1, m_2}}{z_{12}^{\Delta_1 + \Delta_2 - \Delta_3}} G_{-1/2}^{-, su} \mathcal{V}_{m_3}^{(1)} G_{-1/2}^{+, sl} \mathcal{V}_{m_3}'^{(1)}. \quad (4.21)$$

In the case  $m_1 + m_2 = 0$  the product of the corresponding cohomologies yields the  $\mathcal{O}_{-\frac{k}{2}}$

$$\mathcal{O}_{-\frac{k}{2}}^{(1)} = \mathcal{O}_{\frac{k}{2}}^{(1)} = c \partial \xi^+ \mathcal{V}_{-\frac{k}{2}}^{(1)} e^{-2\phi_-} e^{-\phi_+} + \partial \xi^- c \mathcal{V}_{\frac{k}{2}}^{(1)} e^{-2\phi_+} e^{-\phi_-}. \quad (4.22)$$

One further finds that the operator  $\mathcal{O}_{-\frac{k}{2}}^{(1)}$  acts (up to an insignificant numerical constant) as the identity operator within the ground ring. The technical details related to the fusion of operators  $\mathcal{O}_m^{(1)}$  appear in appendix C.

It is not hard to see from the above OPEs that the operators  $\mathcal{O}_m^{(1)}$  generate a cohomology subring isomorphic to  $\mathbb{Z}_k$  with a generator  $x = \mathcal{O}_{1-k/2}^{(1)}$ .<sup>1</sup>

## 5 Ground ring operators at level two

In this section we construct the ground ring operators at level two. As we show below these operators play an important role in constructing the complete ground ring. We start by computing the null vector belonging to superconformal family generated by  $|\Delta, q\rangle$ , where  $\Delta$  and  $q$  satisfy relation (3.1) with  $r = 2$  and  $s = 1$

$$\Delta = \frac{q^2}{2(\tilde{c} - 1)} + \frac{1}{8}(\tilde{c} - 1) - \frac{\tilde{c}^2}{2(\tilde{c} - 1)}. \quad (5.1)$$

From the Kac determinant we know that this family has a null vector at level two with the R-charge equal to the original one. Hence we can write down the following ansatz for the null vector

$$\hat{f}^{(2)}|\Delta, q\rangle = (AL_{-2} + BJ_{-2} + CL_{-1}^2 + DJ_{-1}^2 + EL_{-1}J_{-1} + FG_{-1/2}^+ G_{-3/2}^- + gG_{-3/2}^+ G_{-1/2}^- + HL_{-1}G_{-1/2}^+ G_{-1/2}^- + IJ_{-1}G_{-1/2}^+ G_{-1/2}^-)|\Delta, q\rangle. \quad (5.2)$$

---

<sup>1</sup>Note that to compute powers of  $x$  one only needs to use the product rule (4.17).

It is a straightforward exercise to compute the coefficients in this expression. We find (up to an overall normalization)

$$\begin{aligned}
A &= 1 \\
B &= \frac{4 + 3q - \tilde{c}(2 + q)}{(-1 + \tilde{c})^2} \\
C &= \frac{2}{-1 + \tilde{c}} \\
D &= -\frac{(-3 + \tilde{c} - 2q)(1 + \tilde{c} + 2q)}{2(-1 + \tilde{c})^3} \\
E &= -\frac{4(1 + q)}{(-1 + \tilde{c})^2} \\
F &= \frac{2(-3 + \tilde{c})\tilde{c} - 4q}{(-1 + \tilde{c})^2(-3 + \tilde{c} + 2q)} \\
g &= -\frac{2(1 + \tilde{c} + 2q)((-3 + \tilde{c})\tilde{c} + 2q)}{(-1 + \tilde{c})^2(1 + \tilde{c} - 2q)(-3 + \tilde{c} + 2q)} \\
H &= -\frac{16q}{(-1 + \tilde{c})(1 + \tilde{c} - 2q)(-3 + \tilde{c} + 2q)} \\
I &= \frac{16(1 + \tilde{c} + 2\Delta)}{(-1 + \tilde{c})(1 + \tilde{c} - 2q)(-3 + \tilde{c} + 2q)} .
\end{aligned} \tag{5.3}$$

Following the general procedure outlined in section 3, we would like to find an operator such that the action of BRST operator yields a null physical operator at ghost number one

$$\{Q, \mathcal{O}_m^{(2)}\} = ce^{-\phi_+}e^{-\phi_-}(\lambda \hat{f}^{(2)(su)} + \lambda' \hat{f}^{(2)(sl)})\mathcal{V}_m^{(2)}\mathcal{V}'_m^{(2)} . \tag{5.5}$$

In this expression  $\mathcal{V}_m^{(2)}$  and  $\mathcal{V}'_m^{(2)}$  are  $SU(2)/U(1)$  and  $SL(2)/U(1)$  degenerate operators at level 2

$$\mathcal{V}_m^{(2)}\mathcal{V}'_m^{(2)} = \mathcal{V}_{\frac{k-3}{2},m}\mathcal{V}'_{-\frac{k+1}{2},m} \equiv \mathcal{V}\mathcal{V}' . \tag{5.6}$$

Now we are ready to write down an ansatz for the ground ring operator

$$\begin{aligned}
\mathcal{O}^{(2)} = & c\partial\tilde{b}\mathcal{V}\mathcal{V}'e^{-\phi_+}e^{-\phi_-} + a_0cb\mathcal{V}\mathcal{V}'e^{-\phi_+}e^{-\phi_-} + \\
& c\partial^2\xi^+\{a_1^{(-)}G_{-1/2}^{-(su)} + b_1^{(-)}G_{-1/2}^{-(sl)}\}\mathcal{V}\mathcal{V}'e^{-\phi_+}e^{-2\phi_-} + \\
& c\partial\xi^+\partial\phi_-\{a_{11}^{(-)}G_{-1/2}^{-(su)} + b_{11}^{(-)}G_{-1/2}^{-(sl)}\}\mathcal{V}\mathcal{V}'e^{-\phi_+}e^{-2\phi_-} + \\
& c\partial\xi^+\{a_2^{(-)}L_{-1}^{(su)}G_{-1/2}^{-(su)} + b_2^{(-)}L_{-1}^{(sl)}G_{-1/2}^{-(sl)} + c_2^{(-)}L_{-1}^{(su)}G_{-1/2}^{-(sl)} + d_2^{(-)}L_{-1}^{(sl)}G_{-1/2}^{-(su)} \\
& a_3^{(-)}J_{-1}^{(su)}G_{-1/2}^{-(su)} + b_3^{(-)}J_{-1}^{(sl)}G_{-1/2}^{-(sl)} + c_3^{(-)}J_{-1}^{(su)}G_{-1/2}^{-(sl)} + d_3^{(-)}J_{-1}^{(sl)}G_{-1/2}^{-(su)} + \\
& a_4^{(-)}G_{-3/2}^{-(su)} + b_4^{(-)}G_{-3/2}^{-(sl)} + \}\mathcal{V}\mathcal{V}'e^{-\phi_+}e^{-2\phi_-} + \\
& \{terms\ obtained\ by\ replacing\ +\ superscript\ by\ -\} + \\
& \{a_5e^{\alpha_5\tilde{c}\tilde{b}}L_{-1}^{(su)} + b_5e^{\beta_5\tilde{c}\tilde{b}}L_{-1}^{(sl)} + a_6e^{\alpha_6\tilde{c}\tilde{b}}J_{-1}^{(su)} + b_6e^{\beta_6\tilde{c}\tilde{b}}J_{-1}^{(sl)} + \\
& a_7e^{\alpha_7\tilde{c}\tilde{b}}G_{-1/2}^{+(su)}G_{-1/2}^{-(su)} + b_7e^{\beta_7\tilde{c}\tilde{b}}G_{-1/2}^{+(sl)}G_{-1/2}^{-(sl)} + \\
& c_7e^{\gamma_7\tilde{c}\tilde{b}}G_{-1/2}^{+(su)}G_{-1/2}^{-(sl)} + d_7e^{\delta_7\tilde{c}\tilde{b}}G_{-1/2}^{+(sl)}G_{-1/2}^{-(su)}\}\mathcal{V}\mathcal{V}'e^{-\phi_+}e^{-\phi_-} .
\end{aligned} \tag{5.7}$$

Using the form of the null vector (5.3) and the BRST current (A.5) in equation (5.5) one can find the ground ring element. The result of this calculation is quite cumbersome for general  $m$  and we will not present it here. Instead we note that one can generate all level two ground ring operators using the relation

$$PCO^+\mathcal{O}^{(2)}_{-\frac{k-3}{2}}PCO^-\mathcal{O}_m^{(1)} = \mathcal{O}_{m+\frac{3}{2}}^{(2)} , \tag{5.8}$$

which can be derived using (B.23) (and corresponding expression for the  $SU(2)$  part). Hence for all applications it is sufficient to know the form of the  $\mathcal{O}_m^{(2)}$  for special value of  $m = -(k-3)/2$ . In this case the  $SU(2)/U(1)$  operator  $\mathcal{V}$  becomes chiral, which simplifies the result considerably. The application of  $PCO^+$  on  $\mathcal{O}^{(2)}$  yields then the following result<sup>1</sup>

$$\begin{aligned}
PCO^+\mathcal{O} = & c\partial\xi^+(2(1+k)L_{-1}^{(su)} + (1-k)L_{-1}^{(sl)} - 2kJ_{-1}^{(su)} + 2kJ_{-1}^{(sl)} + \frac{1}{2}G_{-1/2}^{+(sl)}G_{-1/2}^{-(su)} \\
& + \frac{k+1}{2}G_{-1/2}^{+(sl)}G_{-1/2}^{-(sl)})\mathcal{V}\mathcal{V}'e^{-\phi^+}e^{-\phi^-} + \\
& \frac{1}{2}c\partial\xi^-(G_{-3/2}^{+(sl)}G_{-1/2}^{+(sl)} + (k+1)G_{-3/2}^{+(su)}G_{-1/2}^{+(sl)})\mathcal{V}\mathcal{V}'e^{-2\phi^+} + \\
& (2G_{-3/2}^{+(su)} + 2(1+k)G_{-3/2}^{+(sl)} + kL_{-1}^{(sl)}G_{-1/2}^{+(sl)} - kL_{-1}^{(su)}G_{-1/2}^{+(sl)} - \\
& (1+2k)J_{-1}^{(sl)}G_{-1/2}^{+(sl)} + (k-1)J_{-1}^{(su)}G_{-1/2}^{+(sl)})\mathcal{V}\mathcal{V}'e^{-\phi^+} + \\
& \tilde{c}\tilde{b}(-2kG_{-3/2}^{+(su)} + 2kG_{-3/2}^{+(sl)} + (k-1)L_{-1}^{(sl)}G_{-1/2}^{+(sl)} - (1+2k)L_{-1}^{(su)}G_{-1/2}^{+(sl)} - \\
& 2kJ_{-1}^{(sl)}G_{-1/2}^{+(sl)} + 2kJ_{-1}^{(su)}G_{-1/2}^{+(sl)})\mathcal{V}\mathcal{V}'e^{-\phi^+} .
\end{aligned} \tag{5.9}$$

We expect from the fusion rules of the level two operators that

$$PCO^+\mathcal{O}^{(2)}PCO^-\mathcal{O}^{(2)} \sim \mathcal{O}^{(1)} + \mathcal{O}^{(3)} . \tag{5.10}$$

Hence using this relation one can generate all the f-series elements of the ground ring.

<sup>1</sup>We use Mathematica to solve (5.5) and find the coefficients in (5.9).

## 6 BRST cohomologies related to the $g$ -series

As was already noted in section 3, the construction of ground ring elements which correspond to null vectors in the  $g$ -series (3.2) is complicated by the fact that there is no simple general ansatz that would be similar to (3.4). Below we will analyze some particular ground ring operators based on the  $g$ -series. Those examples show that in the Verma module those operators  $\mathcal{O}$  satisfy an equation  $Q_{BRST}\mathcal{O} = \mathcal{O}'$ . Here  $\mathcal{O}'$  is a null operator which in general is given by a nontrivial linear combination of *descendants* of some primitive null vectors. This set of primitive null vectors can contain vectors from the  $g$ -series as well as from the  $f$ -series. This is to be contrasted with the simple equation (3.8).

### The identity operator

We have in the matter sector

$$Q_{BRST}|0\rangle = cL_{-1}|0\rangle + \gamma^- G_{-1/2}^+|0\rangle + \gamma^+ G_{-1/2}^-|0\rangle. \quad (6.1)$$

The identity operator has  $q = R = 0$ ,  $\Delta = 0$ . It belongs to the  $g$ -series of degenerate operators and has two null vectors at level 1/2:  $G_{-1/2}^\pm|0\rangle$ . The vector  $L_{-1}|0\rangle$  is a sum of descendants of these two null vectors. This follows from the anticommutator  $\{G_{-1/2}^+, G_{-1/2}^-\} = 2L_{-1}$ .

### Operators $\mathcal{S}^\pm$

Consider the operators  $\mathcal{S}^\pm$  (3.9) which are ground ring elements in the Ramond sector. The operators  $\mathcal{V}_{0,0}(RR, \pm)$  and  $\mathcal{V}'_{0,0}(RR, \pm)$  belong to the  $g$ -series. The operators

$$G_{-1}^\pm \mathcal{V}_{0,0}(RR, \pm) \mathcal{V}'_{0,0}(RR, \pm), \quad G_0^\mp \mathcal{V}_{0,0}(RR, \pm) \mathcal{V}'_{0,0}(RR, \pm). \quad (6.2)$$

are null.

When we check the BRST invariance of the operators  $\mathcal{S}^\pm$  we obtain

$$\begin{aligned} [Q_{BRST}, \mathcal{S}^\pm] = & e^{\pm \bar{b}c/2} [\eta^\pm e^{3\phi_\pm/2 - \phi_\mp/2} G_{-1}^\mp + \eta^\mp e^{\phi_+/2 + \phi_-/2} G_0^\pm + \\ & c e^{\pm(\phi_+/2 - \phi_-/2)} (L_{-1} \pm \frac{1}{2} J_{-1})] \mathcal{V}_{0,0}(RR, \mp) \mathcal{V}'_{0,0}(RR, \mp). \end{aligned} \quad (6.3)$$

While the first two operators in the right hand side of (6.3) are primitive null operators (6.2) the operators

$$(L_{-1} \pm \frac{1}{2} J_{-1}) \mathcal{V}_{0,0}(RR, \mp) \mathcal{V}'_{0,0}(RR, \mp) \quad (6.4)$$

can be represented as sums of descendants of the null vectors (6.2). This follows from the anticommutators

$$\{G_0^+, G_{-1}^-\} = 2L_{-1} + J_{-1} \quad \{G_{-1}^+, G_0^-\} = 2L_{-1} - J_{-1}. \quad (6.5)$$

## Operators $(\mathcal{S}^\pm)^2$

Consider the operators  $(\mathcal{S}^\pm)^2$  given in (3.12). The primaries

$$\mathcal{V}_\pm \equiv \mathcal{V}_{\frac{k-2}{2}, \pm \frac{k-2}{2}}, \quad \mathcal{V}'_\pm \equiv \mathcal{V}'_{\frac{k+2}{2}, \pm \frac{k+2}{2}} \quad (6.6)$$

belong simultaneously to the  $g$ -series where they have a null vector at level  $1/2$  and to the  $f$ -series where they have a null vector at level 1. The corresponding null operators read

$$\begin{aligned} & G_{-1/2}^\pm \mathcal{V}_\mp, \quad G_{-1/2}^\pm \mathcal{V}'_\pm, \\ & [2(\frac{1}{k} - 1)(L_{-1} \pm J_{-1}) + G_{-1/2}^\pm G_{-1/2}^\mp] \mathcal{V}_\pm, \\ & [-2(\frac{1}{k} + 1)(L_{-1} \pm J_{-1}) + G_{-1/2}^\pm G_{-1/2}^\mp] \mathcal{V}'_\mp. \end{aligned} \quad (6.7)$$

Note that the  $f$ -series vectors are up to the  $1/2$ -descendants of the  $g$ -series vectors just  $(L_{-1} \pm J_{-1})\mathcal{V}_\pm, (L_{-1} \pm J_{-1})\mathcal{V}'_\mp$ .

We have

$$\begin{aligned} Q_{BRST}(\mathcal{S}^\pm)^2 &= e^{\pm \bar{b}c} [\eta^\pm e^{2\phi_\pm - \phi_\mp} G_{-3/2}^\mp \mathcal{V}_\pm \mathcal{V}'_\mp + (\partial\eta^\pm + \partial\phi_\pm \eta^\pm) e^{2\phi_\pm - \phi_\mp} G_{-1/2}^\mp \mathcal{V}_\pm \mathcal{V}'_\mp + \\ & c e^{\pm(\phi_+ - \phi_-)} (L_{-1} \pm J_{-1}) \mathcal{V}_\pm \mathcal{V}'_\mp]. \end{aligned} \quad (6.8)$$

The second and the third terms in the RHS are null. This follows directly from the form of the primitive null vectors (6.7). Whereas the first term can be represented as a sum of descendants of the  $g$  and  $f$ -series null vectors (6.7) because of the commutation relations

$$[L_{-1} + J_{-1}, G_{-1/2}^-] = -G_{-3/2}^-, \quad [L_{-1} - J_{-1}, G_{-1/2}^+] = -G_{-3/2}^+. \quad (6.9)$$

Consider now a general situation. From (3.2) we find that any primary  $\mathcal{V}_{j,m}$  belongs to the  $g$ -series and has two null vectors at levels  $l = j + \frac{1}{2} \pm m$  while a primary  $\mathcal{V}'_{h,m'}$  belongs to the  $g$  series if  $l = m' + \frac{1}{2} \pm h \in \frac{1}{2}\mathbb{Z}$ . The corresponding  $SL(2)$  representation based on  $\mathcal{V}'_{h,m'}$  is finite if  $h < 0$ . The condition that the total  $R$ -charge of the operator  $\mathcal{V}_{j,m} \mathcal{V}_{h,m'}$  is an integer  $R$  implies that  $m - m' = \frac{k}{2}R$ . The total dimension of this operator is

$$\Delta = \frac{(j+h)(j-h+1)}{k} + \frac{(m+m')R}{2}. \quad (6.10)$$

Generically for  $\Delta$  to be an integer or a half-integer one has to set  $h+j=0$  or  $h+j=-k/2$  (note the unitarity bounds on  $h$  and  $j$ ). Or alternatively one can have the reflected equations with  $h$  replaced by  $1-h$ . In the remainder of this section we will concentrate on the case  $h=-j$ . (Cohomologies based on the matter primaries with  $h+j=-k/2$  can be obtained by applying the spectral flow to the cohomologies based on the  $f$ -series considered in the previous two sections.) Since  $|m| \leq j, |m'| \leq h$  it follows that in  $m=m'$

in this case. Denote such a primary  $\mathcal{VV}' \equiv \mathcal{V}_{j,m} \mathcal{V}_{-j,m}$ . Its total  $R$ -charge and dimension are zero. The prescribed picture numbers  $\Pi_{\pm} = 0$  and the vanishing total  $R$ -charge allow for an operator of the form

$$\mathcal{O} = F(b, \tilde{b}, c, \tilde{c})(\gamma^+)^{N_+}(\gamma_-)^{N_-}(\beta^+)^{M_+}(\beta_-)^{M_-} \mathcal{VV}'. \quad (6.11)$$

The  $R$ -charge conservation requires  $N_+ - N_- - M_+ + M_- = 0$ . We further note that the combinations  $\gamma^- \beta^+$  and  $\gamma^+ \beta^-$  do not carry any charges except for the conformal weight 1. Since there are no operators in the ghost Fock space with ghost number zero and negative conformal dimension the above two combinations can be dropped. Since  $\beta^{\pm}$  have positive conformal dimension this leaves us with the powers  $(\gamma^+ \gamma^-)^N$ . The latter have ghost number  $2N$  and conformal dimension  $-N$ . One finally notes that there are no combinations of the remaining ghosts  $b, c, \tilde{b}, \tilde{c}$  that can compensate both of those charges. We conclude that  $N = 0$ . The only uncharged combination of the remaining ghosts is  $c\tilde{b}$ . The upshot of these considerations is that the only ghost structure that can be present in a nontrivial cohomology representative based on the  $\mathcal{VV}'$  is  $c\tilde{b}$ . (The above discussion concerned the primaries themselves, by similar considerations one can conclude that no descendant of such a family can be present in a nontrivial cohomology.) By acting with the BRST charge  $Q_{BRST}$  on an operator of the form

$$e^{\alpha c\tilde{b}} \mathcal{VV}' \quad (6.12)$$

we obtain

$$e^{\alpha c\tilde{b}} [\gamma^+ G_{-1/2}^- + \gamma^- G_{-1/2}^+] \mathcal{VV}' + c L_{-1} \mathcal{VV}' - \alpha c J_{-1} \mathcal{VV}'. \quad (6.13)$$

The right hand side of this expression has a zero norm only if  $\mathcal{VV}'$  is the identity operator and  $\alpha = 0$ . Thus we see that (up to the operators obtained by applying the spectral flow to the  $f$ -series based cohomologies) the identity operator is essentially the only cohomology in picture  $(0, 0)$  based on the  $g$ -series.

## 7 Conclusions and future directions

In this paper we studied BRST cohomologies at ghost number zero in the  $\mathcal{N} = 2$  minimal string theory. Such cohomologies are related to null vectors present in the  $SU(2)/U(1)$  and  $SL(2)/U(1)$  parafermion representation spaces on which one builds up the matter sector of the theory. We explicitly constructed the ground ring elements corresponding to level one and level two primitive null states belonging to the  $f$ -series of degenerate representations. The superparafermion fusion rules together with our analysis of the cohomologies based on the  $g$ -series imply that these elements generate all of the ground ring<sup>1</sup>. A particularly simple subring isomorphic to the  $\mathbb{Z}_k$  ring, where  $k$  is the level of

<sup>1</sup>There can still be some accidental cohomologies based on the  $g$ -series at higher pictures, but we do not think those can be of much practical value



parafermion algebras, is generated by the cohomologies based on the degenerate operators with the null states at level one. This construction is described in detail in section 4 of this paper.

As mentioned in the introduction, the study of the ground ring in the  $\mathcal{N} = 2$  minimal string theory presented in this paper is motivated by a number of potential applications. One of them is the structure of topological amplitudes on the CHS background. To be able to apply the ground ring in this context one needs to investigate the structure of the module over the ground ring that is carried by the physical operators with ghost number one. Possible contact terms in the correlators also need to be understood. We hope to address these issues and to compute some amplitudes explicitly by this method in the future.

Another interesting direction is extending the interrelations between D-branes, the ground ring and a spectral curve found in [3] to the  $\mathcal{N} = 2$  minimal strings. It was found in [3] that the geometrical object behind minimal bosonic and minimal type 0 strings is a complex curve parameterized by a complexified boundary Liouville constant. The singularities of this curve are in correspondence with the Liouville theory ZZ-branes [4] on the one hand and with the ground ring and its module relations on the other. D-branes in  $\mathcal{N} = 2$  super Liouville theory were studied in a number of papers [26–29]. The relations between the  $\mathcal{N} = 2$  analogs of the ZZ-branes found in [26] and their FZZT-brane [30] counterparts suggest that in the  $\mathcal{N} = 2$  case there is a complex surface analogous to the spectral curve of [3]. We will report on the details of its construction and the relations with the ground ring elsewhere [31]. Elucidating this structure potentially may be useful for finding a matrix model description of the  $\mathcal{N} = 2$  two-dimensional black hole and little strings.

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## A $\mathcal{N} = 2$ superconformal ghosts

$$\begin{aligned}
T^{gh} &= 2\partial cb + c\partial b + \partial\tilde{c}\tilde{b} - \frac{3}{2}(\partial\gamma^-\beta^+ + \partial\gamma^+\beta^-) - \frac{1}{2}(\gamma^-\partial\beta^+ + \gamma^+\partial\beta^-), \\
G_{gh}^\pm &= -2\gamma^\pm b \pm 2\partial\gamma^\pm\tilde{b} \pm 2\gamma^\pm\partial\tilde{b} + \frac{3}{2}\partial c\beta^\pm + c\partial\beta^\pm \pm \tilde{c}\beta^\pm, \\
J^{gh} &= \partial(c\tilde{b}) + \gamma^-\beta^+ - \gamma^+\beta^-.
\end{aligned} \tag{A.1}$$

These operators satisfy  $N = 2$  algebra [32]. The super ghosts OPE reads

$$\beta^\pm(z)\gamma^\mp(w) \sim \frac{-1}{z-w} \quad (\text{A.2})$$

We bosonise superconformal ghosts as

$$\beta^\pm \cong e^{-\phi_\mp} \partial \xi^\pm; \quad \gamma^\pm \cong \eta^\pm e^{\phi_\pm} \quad (\text{A.3})$$

which implies

$$\eta^\pm(z)\xi^\mp(0) \sim \frac{1}{z} \quad (\text{A.4})$$

In (A.3)  $\phi$  is the canonically normalized scalar. It is sometimes convenient to express the BRST current in terms of bosonised ghosts

$$\begin{aligned} J_{BRST} = & cT + \eta^+ e^{\phi^+} G^- + \eta^- e^{\phi^-} G^+ + \tilde{c}J + \\ & c[\partial c b + \partial \tilde{c} \tilde{b} - \frac{1}{2}(\partial \phi^+)^2 - \frac{1}{4}\partial^2 \phi^+ - \frac{1}{2}(\partial \phi^-)^2 - \frac{1}{4}\partial^2 \phi^- - \eta^+ \partial \xi^- - \eta^- \partial \xi^+] \\ & - 2\eta^+ e^{\phi^+} \eta^- e^{\phi^-} b - \partial(\eta^- e^{\phi^-}) \eta^+ e^{\phi^+} \tilde{b} + \partial(\eta^+ e^{\phi^+}) \eta^- e^{\phi^-} \tilde{b} \\ & \frac{3}{4}\partial c(\partial \phi^+ + \partial \phi^-) + \tilde{c}(\partial \phi^- - \partial \phi^+) \end{aligned} \quad (\text{A.5})$$

Below are expressions for BRST variations of certain operators in the ghost CFT

$$\begin{aligned} \{Q, c\} &= c\partial c - 2\gamma^- \gamma^+ \\ \{Q, \gamma^\pm\} &= c\partial \gamma^\pm - \frac{1}{2}\gamma^\pm \pm \tilde{c}\gamma^\pm \\ \{Q, \tilde{c}\} &= c\partial \tilde{c} + 2\gamma^- \partial \gamma^+ - 2\gamma^+ \partial \gamma^- \\ \{Q, \tilde{b}c\} &= 2\gamma^- \gamma^+ \tilde{b} + cJ - c(\partial \phi^+ - \partial \phi^-) \\ \{Q, e^{-\phi^+} e^{-\phi^-}\} &= \partial c e^{-\phi^+} e^{-\phi^-} + c\partial(e^{-\phi^+} e^{-\phi^-}) \\ \{Q, e^{\phi^+} e^{-\phi^-}\} &= -\partial c e^{\phi^+} e^{-\phi^-} + c\partial(e^{\phi^+} e^{-\phi^-}) + \eta_+ e^{2\phi^+} e^{-\phi^-} G^- + 2\tilde{c} e^{\phi^+} e^{\phi^-} + 2\eta^- \eta^+ e^{2\phi^+} \tilde{b} \end{aligned} \quad (\text{A.6})$$

## B Superparafermions and their fusion rules

Here we summarize some basic facts on the  $SL(2)/U(1)$  and  $SU(2)/U(1)$  superparafermions introduced in section 2. (The two theories are closely related, and most of the formulae differ by the substitution  $k \rightarrow -k$ ). In the following we will suppress antiholomorphic terms. The R-current of  $\mathcal{N} = 2$  algebra of superparafermions can be bosonised as

$$J_R = i\sqrt{\frac{c}{3}} \partial X_R. \quad (\text{B.1})$$

The expressions for  $\mathcal{N} = 2$  primaries in the R sector are [20]

$$\Delta[V_{h;m}^{(sl,susy)}(R, \pm)] = -\frac{h(h-1)}{k} + \frac{(m \pm \frac{1}{2})^2}{k} + \frac{1}{8} \quad (\text{B.2})$$

and

$$q_{sl,R} = \pm \frac{1}{2} + \frac{2m \pm 1}{k} \quad (\text{B.3})$$

The corresponding  $SU(2)/U(1)$  formulae are

$$\Delta[V_{h;m}^{(su,susy)}(R, \pm)] = \frac{j(j+1)}{k} - \frac{(m \pm \frac{1}{2})^2}{k} + \frac{1}{8} \quad (\text{B.4})$$

and

$$q_{su,R} = \pm \frac{1}{2} - \frac{2m \pm 1}{k} \quad (\text{B.5})$$

The expressions for the conformal dimensions and R-charges of parafermion vertex operators in the NS sector are given by (2.8)–(2.11). The superconformal primaries  $\mathcal{V}'_{h,m}$  are related to the primaries of the bosonic  $SL(2)/U(1)$  theory (see e.g. [20])

$$\mathcal{V}'_{h,m} = V_{h;m}^{(sl)} e^{i\alpha_m X_R}, \quad \alpha_m = \frac{2m}{\sqrt{k(k+2)}} \quad (\text{B.6})$$

Primaries of the bosonic  $SL(2)/U(1)$  theory are related to the primaries of  $SL(2)_{k-2}$  WZW model  $\Phi_h(x, \bar{x})$  through the coupling to the extra  $U(1)$  whose bosonised current we denote by  $\partial Y$ . In addition, there is an analog of Fourier transform, which diagonalizes  $\Phi_h(x, \bar{x})$  in the  $J_3^{sl}$  basis (eigenvalue of  $J_3^{sl}$  is conventionally denoted by  $m$ ).

$$V_{h;m,\bar{m}}^{(sl)} = e^{i\sqrt{\frac{2}{k}}(mY - \bar{m}\bar{Y})} \int d^2x x^{h+m-1} \bar{x}^{h+\bar{m}-1} \Phi_h(x, \bar{x}) \quad (\text{B.7})$$

The construction of  $SU(2)/U(1)$  parafermions is very similar and can be found for example in [33, 34]

The  $SU(2)$  fusion rules are

$$[\mathcal{V}_{\frac{k-2}{2},m}][\mathcal{V}_{\frac{k-2}{2},m'}] \sim [\mathcal{V}_{0,m+m'}] = \begin{cases} [G_{-1/2}^- \mathcal{V}_{\frac{k-2}{2},m+m'-k/2}] & \text{if } k-2 > m+m' > 0 \\ [G_{-1/2}^+ \mathcal{V}_{\frac{k-2}{2},m+m'+k/2}] & \text{if } 2-k < m+m' < 0 \\ 1 & \text{if } m+m' = 0 \end{cases} \quad (\text{B.8})$$

while in  $SL(2)$  we have [26, 27]

$$[\mathcal{V}'_{-k/2,m}][\mathcal{V}'_{-k/2,m'}] \sim [G_{-1/2}^- \mathcal{V}'_{-k/2,m+m'+k/2}] + [\mathcal{V}'_{-k,m+m'}] + [\mathcal{V}'_{0,m+m'}] \quad (\text{B.9})$$

$$\text{if } 0 > m+m' > -(k-2).$$

For integral level  $k$ , that is the case of interest for the minimal  $N=2$  strings, a special degeneration occurs in the  $SL(2)/U(1)$  OPE B.9 - the family  $[\mathcal{V}'_{-k,m+m'}]$  drops out of the theory (see e.g. [35]).

In general if we have an OPE of the type

$$\mathcal{V}'_1(z) \mathcal{V}'_2(0) \sim CG_{-1/2}^- \mathcal{V}'_3(0). \quad (\text{B.10})$$

with the fusion coefficient  $C$ , it turns out that the fusion coefficients in the following OPE's

$$\begin{aligned} G_{-1/2}^+ \mathcal{V}'_1(z) \mathcal{V}'_2(0) &\sim C_1 \mathcal{V}'_3(0) \\ \mathcal{V}'_1(z) G_{-1/2}^- \mathcal{V}'_2(0) &\sim C_2 \mathcal{V}'_3(0), \end{aligned} \quad (\text{B.11})$$

are

$$C_1 = -C_2 = -C(2\Delta_3 + q_3). \quad (\text{B.12})$$

We will also need to know the fusion coefficients in the following OPE

$$G_{1/2}^- \mathcal{V}'_1(z) \mathcal{V}'_2(0) \sim C_3 G_{-3/2}^- G_{-1/2}^- \mathcal{V}'_3(0) \quad (\text{B.13})$$

To compute it we consider the following 3-pt function <sup>1</sup>

$$\begin{aligned} \langle G_{-1/2}^- \mathcal{V}'_1 \mathcal{V}'_2 G_{-3/2}^+ G_{-1/2}^+ \mathcal{V}'_3 \rangle &= \langle G_{-1/2}^- \mathcal{V}'_1 G_{-1/2}^+ \mathcal{V}'_2 G_{-1/2}^+ \mathcal{V}'_3 \rangle \left( \frac{1}{z_1 - z_3} - \frac{1}{z_2 - z_3} \right) - \\ &\frac{(2\Delta_1 + q_1)}{(z_1 - z_3)^2} \langle \mathcal{V}'_1 \mathcal{V}'_2 G_{-1/2}^+ \mathcal{V}'_3 \rangle \end{aligned} \quad (\text{B.14})$$

This expression allows us to find also the following fusion coefficient

$$G_{-1/2}^- \mathcal{V}'_1 G_{-1/2}^+ \mathcal{V}'_2 \sim C_4 G_{-1/2}^- \mathcal{V}'_3 \quad (\text{B.15})$$

We find

$$\begin{aligned} C_4 + C(2\Delta_1 + q_1) &= 0 \\ C_3(2\Delta_3 + 3q_3 - 2 + 2\tilde{c}) &= -C(2\Delta_1 + q_1) \end{aligned} \quad (\text{B.16})$$

The expressions above are derived for the fusion rules (B.9) which are relevant for the  $SL(2)/U(1)$  parafermions. Similar expressions hold for their  $SU(2)/U(1)$  counterparts; one is instructed to invert the sign of the  $U(1)_R$  charge.

There is an alternative way to derive the fusion rules and the structure constants discussed above. It involves considering the null vectors and their effect on the OPEs. Let's start with the simplest case of level one null vector. As discussed above the null vector has the following form

$$[(q_1 - 1)L_{-1} - (2\Delta_1 + 1)J_{-1} + G_{-1/2}^+ G_{-1/2}^-] |\Delta_1, q_1\rangle \equiv \hat{f} |\Delta_1, q_1\rangle, \quad (\text{B.17})$$

where  $\Delta_1$  and  $q_1$  satisfy

$$2(\tilde{c} - 1)\Delta_1 = q_1^2 - \tilde{c}. \quad (\text{B.18})$$

We will be interested in OPE's of the following type (we suppress the  $z$  dependence)

$$\mathcal{V}'_1(z) \mathcal{V}'_2(0) \sim C G_{-1/2}^+ \mathcal{V}'_3(0) \quad (\text{B.19})$$

$$G_{-1/2}^- \mathcal{V}'_1(z) G_{-1/2}^+ \mathcal{V}'_2(0) \sim C_4 G_{-1/2}^- \mathcal{V}'_3(0)$$

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<sup>1</sup>Note that  $G_{-3/2}^- G_{-1/2}^- \mathcal{V}'_3$  is a Virasoro primary and hence has simple scaling behavior at infinity.

To find the allowed  $\mathcal{V}'_3$  we consider the following correlator

$$\langle \hat{f} \mathcal{V}'_1(z_1) \mathcal{V}'_2(z_2) G_{-1/2}^+ \tilde{\mathcal{V}}'_3 \rangle = 0, \quad (\text{B.20})$$

where  $\tilde{\mathcal{V}}'_3$  is the conjugate of  $\mathcal{V}'_3$ . Expanding (B.20) we find

$$\begin{aligned} & \left[ (q_1 - 1) \partial_{z_1} - (2\Delta_1 + 1) \left( -\frac{q_2}{z_{21}} + \frac{q_1 + q_2}{z_{31}} \right) \right] \langle \mathcal{V}'_1(z_1) \mathcal{V}'_2(z_2) G_{-1/2}^+ \tilde{\mathcal{V}}'_3 \rangle = \\ & - \langle G_{-1/2}^- \mathcal{V}'_1(z_1) G_{-1/2}^+ \mathcal{V}'_2(z_2) G_{-1/2}^+ \tilde{\mathcal{V}}'_3 \rangle, \end{aligned} \quad (\text{B.21})$$

where  $z_{ij} = z_i - z_j$ . Using this equation we find the following relations

$$\begin{aligned} (q_1 - 1) \left( \Delta_{21} - \frac{1}{2} \right) C &= -(2\Delta_1 + 1) q_2 C + C_4 \\ (\Delta_{31} + \frac{1}{2}) (q_1 - 1) &= (2\Delta_1 + 1) (q_1 + q_2), \end{aligned} \quad (\text{B.22})$$

where  $\Delta_{21} = \Delta_2 + \Delta_1 - \Delta_3$  and  $\Delta_{31} = \Delta_3 + \Delta_1 - \Delta_2$ . One can check that these equations are in complete agreement with (B.9) and (B.16). In terms of  $h, m$  quantum numbers (B.19) can be written as follows

$$\mathcal{V}'_{-\frac{k}{2}, m_1} \mathcal{V}'_{h, m_2} \sim C G_{-1/2}^- \mathcal{V}'_{h, m_1 + m_2 + k/2}. \quad (\text{B.23})$$

The coefficient  $C$  in (B.19) will become zero for  $k$  and  $k - m_1$  integer, since two additional null vectors associated with the zeros of polynomial (3.2) will appear. This problem can be dealt with by starting from irrational  $k$  and redefining the  $SL(2)/U(1)$  operators in such a way to make  $C$  finite.

### C Fusion of the $\mathcal{O}_m^{(1)}$ operators.

Here we give details of the fusion of operators  $\mathcal{O}_m^{(1)}$  constructed in section 4. We start by fixing the normalization of the level one  $SL(2)/U(1)$  operators. Consider the OPE involving the operator  $\mathcal{V}_m^{(1)}$ , with  $m \in \mathbb{R}$  and a generic operator  $\mathcal{V}'_{h, m'}$ ,

$$\begin{aligned} \mathcal{V}_m^{(1)}(z) \mathcal{V}'_{h, m'}(0) &\sim C_{\uparrow} \left( h - m - m' + \frac{k}{2} \right)^{-1} G_{-1/2}^+ \mathcal{V}'_{h, m + m' - \frac{k}{2}} + \\ &C_{\downarrow} \left( h + m + m' + \frac{k}{2} \right)^{-1} G_{-1/2}^- \mathcal{V}'_{h, m + m' + \frac{k}{2}} + \dots, \end{aligned} \quad (\text{C.1})$$

where we suppressed the  $z$ -dependence and the ellipses stand for two additional terms in the OPE, which are not relevant for our consideration. The structure constants  $C_{\uparrow}$  and  $C_{\downarrow}$  can be found in e.g. in [27]. Up to an uninteresting overall multiplicative factors these

are

$$C_{\uparrow}^2 = \frac{\Gamma(1-h-m')\Gamma(1+\frac{k}{2}-m)\Gamma(h+m+m'-\frac{k}{2}-1)}{\Gamma(h+m')\Gamma(-\frac{k}{2}+m)\Gamma(2-h-m-m'+\frac{k}{2})} \quad (C.2)$$

$$C_{\downarrow}^2 = \frac{\Gamma(1-h+m')\Gamma(1+\frac{k}{2}+m)\Gamma(h-m-m'-\frac{k}{2}-1)}{\Gamma(h-m')\Gamma(-\frac{k}{2}-m)\Gamma(2-h+m+m'+\frac{k}{2})} \quad (C.3)$$

We immediately see that both  $C_{\uparrow}$  and  $C_{\downarrow}$  vanish for

$$m = -\frac{k}{2}, -\frac{k}{2} + 1, \dots, \frac{k}{2}, \quad (C.4)$$

as expected. Redefining the  $SL(2)/U(1)$  operators as

$$\tilde{\mathcal{V}}'_m \equiv \lim_{\epsilon \rightarrow 0} \mathcal{V}'_{m+\epsilon} \sqrt{\Gamma\left(-\frac{k}{2} + m + \epsilon\right)}, \quad (C.5)$$

we see that both  $\tilde{C}_{\uparrow}$  and  $\tilde{C}_{\downarrow}$  become finite for  $m$  satisfying (C.4). Note that the case of  $h = -k/2$  and  $m + m' = 0$  is special since the coefficients in (C.1) blow up. At this point we need to switch our attention to the whole operator  $\mathcal{O}_m^{(1)}$ . There is an additional factor  $K_{m_1, m_2}$  in the fusion of  $\mathcal{O}_m^{(1)}$ 's

$$PCO^+ \mathcal{O}_{m_1}^{(1)} PCO^- \mathcal{O}_{m_2}^{(1)} = C_{m_1, m_2} K_{m_1, m_2} \mathcal{O}_{m_3}^{(1)}. \quad (C.6)$$

Here  $m_3 = m_1 + m_2 - k/2$  and  $C_{m_1, m_2} \propto C_{\uparrow}(m_1, m_2) \left(h - m_1 - m_2 + \frac{k}{2}\right)^{-1}$  for  $m_1 + m_2 > 0$  and  $m_3 = m_1 + m_2 + k/2$ ,  $C_{m_1, m_2} \propto C_{\downarrow}(m_1, m_2) \left(h + m_1 + m_2 + \frac{k}{2}\right)^{-1}$  for  $m_1 + m_2 < 0$ . For  $m_1 + m_2 = 0$  the corresponding fusion rule is defined as follows. First of all note that the relevant  $SU(2)$  fusion rule has the form

$$[\mathcal{V}_m^{(1)}][\mathcal{V}_{-m}^{(1)}] \sim [1] \quad (C.7)$$

so that both terms in the right hand side of (C.1) contribute to

$$\mathcal{O}_{-\frac{k}{2}}^{(1)} = \mathcal{O}_{\frac{k}{2}}^{(1)} = c\partial\xi^+ \mathcal{V}_{-\frac{k}{2}}'^{(1)} e^{-2\phi_-} e^{-\phi_+} + \partial\xi^- c\mathcal{V}_{\frac{k}{2}}'^{(1)} e^{-2\phi_+} e^{-\phi_-}. \quad (C.8)$$

The relevant proportionality coefficient  $K_{m, -m}$  in this case vanishes precisely in such a way that the product  $C_{m_1, m_2} K_{m_1, m_2}$  becomes finite. The operator (C.8) acts as an identity operator within the ground ring.

## D Action of the picture raising operators on BRST cohomologies

Here we give an example of a kernel and a cokernel of a picture-raising operator (2.7) considered on BRST cohomologies. We begin with an example of a situation when  $PCO^{\pm}$

map BRST non-trivial operators into BRST trivial ones. The simplest example of such phenomenon is the operator

$$\mathcal{O} = ce^{-\phi_-}e^{-\phi_+}\mathbf{1} . \quad (\text{D.1})$$

This operator is mapped into zero by the action of both  $PCO^\pm$ . In general BRST non-trivial operators of the type

$$\mathcal{O} = ce^{-\phi_-}e^{-\phi_+}\mathcal{V}\mathcal{V}' , \quad (\text{D.2})$$

where  $\mathcal{V}$  and  $\mathcal{V}'$  are (anti-)chiral primaries of  $SU(2)/U(1)$  and  $SL(2)/U(1)$  respectively are mapped into zero by the action of  $PCO^+$  ( $PCO^-$ )<sup>1</sup>

Consider now a BRST invariant operator in the standard  $(-1, -1)$  picture

$$\mathcal{O} = ce^{-\phi_-}e^{-\phi_+}\mathcal{V}'\mathcal{V} . \quad (\text{D.3})$$

The operator  $\mathcal{V}$  and  $\mathcal{V}'$  should be primaries of the corresponding  $\mathcal{N} = 2$  algebras, in order for  $\mathcal{O}$  to be BRST closed. Now let us perform picture changing into the  $(0, -1)$  picture. The result is

$$\mathcal{O}' = ce^{-\phi_-}G_{-\frac{1}{2}}^-(\mathcal{V}'\mathcal{V}) . \quad (\text{D.4})$$

In general the BRST cohomologies in this picture have the following form

$$\tilde{\mathcal{O}} = ce^{-\phi_-}\tilde{\mathcal{V}}'\tilde{\mathcal{V}} , \quad (\text{D.5})$$

where  $\tilde{\mathcal{V}}'\tilde{\mathcal{V}}$  satisfies

$$\begin{aligned} G_r^+(\tilde{\mathcal{V}}'\tilde{\mathcal{V}}) &= 0 , \\ G_{r-1}^-(\tilde{\mathcal{V}}'\tilde{\mathcal{V}}) &= 0 \quad r > 0 , \\ \Delta(\tilde{\mathcal{V}}'\tilde{\mathcal{V}}) &= \frac{1}{2} , \\ q((\tilde{\mathcal{V}}'\tilde{\mathcal{V}})) &= -1 . \end{aligned} \quad (\text{D.6})$$

These conditions are indeed satisfied for the  $\mathcal{O}'$ , but it is easy to see that in this picture there are additional BRST invariant operators corresponding to the fields which are antichiral in  $SU(2)/U(1)$  and  $SL(2)/U(1)$  separately. This operator is based on primaries in  $SU(2)/U(1)$  and  $SL(2)/U(1)$  which are labeled the following quantum numbers

$$\begin{aligned} j^{su} &= m^{su} = \frac{k}{2} - h , \\ m^{sl} &= -h , \\ h^{sl} &= h . \end{aligned} \quad (\text{D.7})$$

A simple combinatorial analysis, similar to the one done in section 6 of the paper, shows that one cannot write down an operator with ghost number one, picture  $(-1, -1)$ , vanishing total  $R$ -charge and dimension, which would be based on a matter primary with dimension  $1/2$  and  $R$ -charge  $-1$ . Thus there is no  $PCO^+$  pre-image for operator (D.5).

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<sup>1</sup>It should be noted, however, that in contrast with the case considered in section 2, here the image of the  $PCO^\pm$  is not an exact zero but rather a null state.

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